CLASSIFYING *-HOMOMORPHISMS

JOINT WORK WITH J. GABE, C. SCHAFHAUSER, A. TIKUISIS, AND S. WHITE

José Carrión
TCU

October 13, 2019

ECOAS, Ohio State

INTRODUCTION

Theorem ("Many hands")

C^*-algebras that are unital, simple, separable, nuclear, regular, and satisfy the UCT, are classified by K-theory and traces.

- C^*-analog of the classification of injective von Neumann factors: Murray-von Neumann, Connes, Haagerup.
- No traces: Kirchberg-Phillips (1990s).
- We focus only on the case $T(A) \neq \varnothing$.
- Classifying invariant:

$$\mathrm{Ell}(A) := \Big(K_0(A),\ [1_A]_0,\ K_1(A),\ T(A),\ T(A)\times K_0(A) \to \mathbb{R}\Big)$$

AN EXAMPLE: $C(X) \rtimes G$

Setup: G, a countable discrete group, acting on X, a compact metric space.

When does $C(X) \rtimes G$ satisfy the hypotheses?

AN EXAMPLE: $C(X) \rtimes G$

Setup: G, a countable discrete group, acting on X, a compact metric space.

When does $C(X) \rtimes G$ satisfy the hypotheses?

$C(X) \rtimes G$ is...

- ... unital, since X is compact;

AN EXAMPLE: $C(X) \rtimes G$

Setup: G, a countable discrete group, acting on X, a compact metric space.

When does $C(X) \rtimes G$ satisfy the hypotheses?

$C(X) \rtimes G$ is...

- ...unital, since X is compact;
- ...simple, if the action is free and minimal;

AN EXAMPLE: $C(X) \rtimes G$

Setup: G, a countable discrete group, acting on X, a compact metric space.

When does $C(X) \rtimes G$ satisfy the hypotheses?

$C(X) \rtimes G$ is...

- ...unital, since X is compact;
- ...simple, if the action is free and minimal;
- ...separable, since G is countable and X is metrizable;

AN EXAMPLE: $C(X) \rtimes G$

Setup: G, a countable discrete group, acting on X, a compact metric space.

When does $C(X) \rtimes G$ satisfy the hypotheses?

$C(X) \rtimes G$ is...

- ...unital, since X is compact;
- ...simple, if the action is free and minimal;
- ...separable, since G is countable and X is metrizable;
- ...nuclear, if G is amenable;

an example: $C(X) \rtimes G$

Setup: G, a countable discrete group, acting on X, a compact metric space.

When does $C(X) \rtimes G$ satisfy the hypotheses?

$C(X) \rtimes G$ is...

- ...unital, since X is compact;
- ...simple, if the action is free and minimal;
- ...separable, since G is countable and X is metrizable;
- ...nuclear, if G is amenable;
- ...in the UCT class, if G is amenable;

AN EXAMPLE: $C(X) \rtimes G$

Setup: G, a countable discrete group, acting on X, a compact metric space.

When does $C(X) \rtimes G$ satisfy the hypotheses?

$C(X) \rtimes G$ is…

- …unital, since X is compact;
- …simple, if the action is free and minimal;
- …separable, since G is countable and X is metrizable;
- …nuclear, if G is amenable;
- …in the UCT class, if G is amenable;
- …regular, if X is finite dimensional and G is f.g. nilpotent.

Finite nuclear dimension (Winter-Zacharias)

- $\dim_{\mathrm{nuc}} A$: noncommutative analog of covering dimension
- $\dim_{\mathrm{nuc}} C(X) = \dim X$
- range of $\mathrm{Ell}(-)$ exhausted by C^*-algebras with $\dim_{\mathrm{nuc}} < \infty$

Finite nuclear dimension (Winter-Zacharias)

- $\dim_{\mathrm{nuc}} A$: noncommutative analog of covering dimension
- $\dim_{\mathrm{nuc}} C(X) = \dim X$
- range of $\mathrm{Ell}(-)$ exhausted by C^*-algebras with $\dim_{\mathrm{nuc}} < \infty$

$\mathcal{Z}$-stability: $A \cong A \otimes \mathcal{Z}$

- Jiang-Su algebra $\mathcal{Z}$: ∞-dim'l analog of $\mathbb{C}$.
- $\mathcal{Z} \sim_{KK} \mathbb{C}$; has unique trace.
- $\mathrm{Ell}(A) = \mathrm{Ell}(A \otimes \mathcal{Z})$

Theorem (Castillejos-Evington-Tikuisis-White-Winter)

For A is a unital, simple, separable, nuclear, nonelementary,

$$\dim_{\mathrm{nuc}} A < \infty \quad \Leftrightarrow \quad A \cong A \otimes \mathcal{Z}.$$

Comments:

- "$\Rightarrow$" is due to Winter
- Can remove "unital" from statement: Castillejos-Evington, Tikuisis.
- Conjecturally equivalent to a third condition: *strict comparison.* (True under mild trace hypotheses.)
- Proof developed important technique for handling complicated trace spaces.

THE ORIGINAL ROAD TO CLASSIFICATION (∼2017)

Impossible to summarize decades of work in a slide.

Some recent components:

Classification of “model” algebras

- Gong-Lin-Niu ’15: classified C^*-algebras with a certain internal tracial approximation structure.
- The class exhausts range of $\mathrm{Ell}(-)$.

Realizing the approximations

- Elliott-Gong-Lin-Niu ’15: abstract conditions on a C^*-algebra $\Rightarrow$ concrete tracial approximations of GLN.
- Tikuisis-White-Winter ’17: the abstract conditions are the ones stated in the classification theorem.

We develop an alternate route to classification: beginning with von Neumann algebraic techniques inspired by work of Connes and Haagerup, we extend the *KK*-theoretic techniques recently developed by Schafhauser to prove classification theorems in an abstract setting.

We develop an alternate route to classification: beginning with von Neumann algebraic techniques inspired by work of Connes and Haagerup, we extend the *KK*-theoretic techniques recently developed by Schafhauser to prove classification theorems in an abstract setting.

We will (mostly) ignore the difficulties that arise from non-separability or the lack of a unit.

CLASSIFYING MORPHISMS & ALGEBRAS

Existence and uniqueness into II_1 factors

Theorem (Connes '76)

Injective von Neumann Algebras are AFD.

Corollary

A: nuclear C^*-algebra; M: II_1 factor.

1. *(existence)*
   $\tau \in T(A) \implies \exists *$-hom $\varphi : A \to M$ s.t. $\tau_M \circ \varphi = \tau$.
2. *(uniqueness)*
   $\varphi, \psi : A \to M$ $*$-hom's s.t. $\tau_M \circ \varphi = \tau_M \circ \psi$
   $\implies \varphi \approx_u \psi \quad (\text{in } \|\cdot\|_2)$

Rough scheme

Produce invariant $\mathrm{inv}(-)$ s.t.
(with abstract hypotheses on A, B):

- (existence)
  $\alpha\colon \mathrm{inv}(A) \to \mathrm{inv}(B) \implies \exists\, \varphi\colon A \to B$ s.t. $\mathrm{inv}(\varphi) = \alpha$;
- (uniqueness)
  $\varphi, \psi\colon A \to B$ and $\mathrm{inv}(\varphi) = \mathrm{inv}(\varphi) \implies \varphi \approx_u \psi$.

Intertwining: $\mathrm{inv}(A) \cong \mathrm{inv}(B) \implies A \cong B$.

Rough scheme

Produce invariant $\mathrm{inv}(-)$ s.t.
(with abstract hypotheses on A, B):

- (existence)
  $\alpha\colon\ \mathrm{inv}(A) \to \mathrm{inv}(B) \implies \exists\, \varphi\colon A \to B \text{ s.t. } \mathrm{inv}(\varphi) = \alpha;$
- (uniqueness)
  $\varphi, \psi\colon A \to B$ and $\mathrm{inv}(\varphi) = \mathrm{inv}(\varphi) \implies \varphi \approx_u \psi.$

Intertwining: $\mathrm{inv}(A) \cong \mathrm{inv}(B) \implies A \cong B.$

$\mathrm{inv}(-)$ will be more complicated than $\mathrm{Ell}(-)$.

Also want:

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \text{ yields } \mathrm{inv}(A) \cong \mathrm{inv}(B).$$

Definition

$\underline{K}(A) = \bigoplus_{n=0}^{\infty} K_0(A; \mathbb{Z}/n\mathbb{Z}) \oplus K_1(A; \mathbb{Z}/n\mathbb{Z})$

Can think of $K_i(A; \mathbb{Z}/n\mathbb{Z})$ as $K_i(A \otimes \mathcal{O}_{n+1})$.

Definition

$\underline{K}(A) = \bigoplus_{n=0}^{\infty} K_0(A; \mathbb{Z}/n\mathbb{Z}) \oplus K_1(A; \mathbb{Z}/n\mathbb{Z})$

Can think of $K_i(A; \mathbb{Z}/n\mathbb{Z})$ as $K_i(A \otimes \mathcal{O}_{n+1})$.

Slogan

Can check "closeness" of $KK(\varphi)$ and $KK(\psi)$ by checking that $\underline{K}(\varphi)$ and $\underline{K}(\psi)$ agree on large finite subsets of $\underline{K}(A)$.

Thomsen-Nielsen (early 90s): different proof of Elliott's classification of simple unital $A\mathbb{T}$ algebras. Need refined inv.

Thomsen-Nielsen (early 90s): different proof of Elliott’s classification of simple unital $A\mathbb{T}$ algebras. Need refined inv.

Definition

$$\overline{K}_1^{\mathrm{alg}}(A) := U^\infty(A)/CU^\infty(A)$$

$CU^\infty(A)$ is the *closure* of the commutator subgroup of $U^\infty(A)$.

Thomsen-Nielsen (early 90s): different proof of Elliott's classification of simple unital A$\mathbb{T}$ algebras. Need refined inv.

Definition

$$\overline{K}_1^{\mathrm{alg}}(A) := U^\infty(A)/CU^\infty(A)$$

$CU^\infty(A)$ is the *closure* of the commutator subgroup of $U^\infty(A)$.

$\overline{K}_1^{\mathrm{alg}}(A)$ came up in Thomsen's work on the role of the relationship between and K-theory and traces in classification theory.

Has seen lots of use in classification (e.g. in GLN).

Definition

$$\operatorname{inv}(A) := \left(\underline{K}(A),\;\; \overline{K}_1^{\mathrm{alg}}(A),\;\; \operatorname{Aff} T(A)\right)$$

Definition

$$\operatorname{inv}(A) := \Big(\underline{K}(A),\ \ \overline{K}_1^{\mathrm{alg}}(A),\ \ \operatorname{Aff} T(A)\Big)$$

A *compatible triple* $(\underline{\alpha}, \beta, \gamma)\colon\ \operatorname{inv}(A) \to \operatorname{inv}(E)$ consists of

$$\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(E), \quad \beta\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(E), \quad \gamma\colon\ \operatorname{Aff} T(A) \to \operatorname{Aff} T(E)$$

such that

$$\begin{array}{ccccccc}
K_0(A) & \xrightarrow{\rho_A} & \operatorname{Aff} T(A) & \xrightarrow{\mathrm{Th}_A} & \overline{K}_1^{\mathrm{alg}}(A) & \longrightarrow & K_1(A) \\
\downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha_1} \\
K_0(E) & \xrightarrow{\rho_E} & \operatorname{Aff} T(E) & \xrightarrow{\mathrm{Th}_E} & \overline{K}_1^{\mathrm{alg}}(E) & \longrightarrow & K_1(E)
\end{array}$$

commutes.

Definition

$$\mathrm{inv}(A) := \left(\underline{K}(A),\ \ \overline{K}_1^{\mathrm{alg}}(A),\ \ \mathrm{Aff}\,T(A)\right)$$

A *compatible triple* $(\underline{\alpha}, \beta, \gamma)\colon\ \mathrm{inv}(A) \to \mathrm{inv}(E)$ consists of

$$\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(E), \quad \beta\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(E), \quad \gamma\colon\ \mathrm{Aff}\,T(A) \to \mathrm{Aff}\,T(E)$$

such that

$$\begin{array}{ccccccc} K_0(A) & \xrightarrow{\rho_A} & \mathrm{Aff}\,T(A) & \xrightarrow{\mathrm{Th}_A} & \overline{K}_1^{\mathrm{alg}}(A) & \longrightarrow & K_1(A) \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha_1} \\ K_0(E) & \xrightarrow{\rho_E} & \mathrm{Aff}\,T(E) & \xrightarrow{\mathrm{Th}_E} & \overline{K}_1^{\mathrm{alg}}(E) & \longrightarrow & K_1(E) \end{array}$$

commutes.

$(\underline{\alpha}, \beta, \gamma)$ is *faithful and amenable* if $\gamma^*\left(T(E)\right) \subseteq T(A)$ consists of faithful amenable traces.

Theorem (C-Gabe-Schafhauser-Tikuisis-White)

- *A : sep., exact, UCT*
- *B : sep., $\mathcal{Z}$-stable, strict comparison w.r.t. $T(B)$, $T(B) \neq \emptyset$ & compact*
- *$(\underline{\alpha}, \beta, \gamma)\colon \operatorname{inv}(A) \to \operatorname{inv}(B)$: compatible triple that is faithful and amenable*

Then:

- *$\exists$ full† nuclear *-hom. $\varphi\colon A \to B$ s.t. $\operatorname{inv}(\varphi) = (\underline{\alpha}, \beta, \gamma)$;*
- *this φ is unique up to approx. unitary equivalence.*

Theorem (C-Gabe-Schafhauser-Tikuisis-White)

- *A : sep., exact, UCT*
- *B : sep., $\mathcal{Z}$-stable, strict comparison w.r.t. $T(B)$, $T(B) \neq \varnothing$ & compact*
- *$(\underline{\alpha}, \beta, \gamma)\colon \operatorname{inv}(A) \to \operatorname{inv}(B)$: compatible triple that is faithful and amenable*

Then:

- *$\exists$ full[†] nuclear *-hom. $\varphi\colon A \to B$ s.t. $\operatorname{inv}(\varphi) = (\underline{\alpha}, \beta, \gamma)$;*
- *this φ is unique up to approx. unitary equivalence.*

[†] : $\varphi(a)$ generates B as an ideal $\forall a \neq 0$. Wrinkle with units.

Theorem (C-Gabe-Schafhauser-Tikuisis-White)

- *A : sep., exact, UCT*
- *B : sep., $\mathcal{Z}$-stable, strict comparison w.r.t. $T(B)$, $T(B) \neq \varnothing$ & compact*
- *$(\underline{\alpha}, \beta, \gamma)\colon \operatorname{inv}(A) \to \operatorname{inv}(B)$: compatible triple that is faithful and amenable*

Then:

- *$\exists$ full[†] nuclear *-hom. $\varphi\colon A \to B$ s.t. $\operatorname{inv}(\varphi) = (\underline{\alpha}, \beta, \gamma)$;*
- *this φ is unique up to approx. unitary equivalence.*

[†] : $\varphi(a)$ generates B as an ideal $\forall a \neq 0$. Wrinkle with units.

Classifying algebras

The (unital) C^*-algebras in

$$\left\{ \begin{matrix} \text{alg's satisfying} \\ \text{domain hyp.} \end{matrix} \right\} \bigcap \left\{ \begin{matrix} \text{alg's satisfying} \\ \text{target hyp.} \end{matrix} \right\} \bigcap \left\{ \begin{matrix} A : \mathrm{id}_A \text{ satisfies} \\ \text{morphism hyp.} \end{matrix} \right\}$$

are precisely those we wanted to classify on first slide.

$\Rightarrow$ Classification of algebras via $\mathrm{inv}(-)$.
Can deduce classification of algebras via $\mathrm{Ell}(-)$.

The (unital) C^*-algebras in

$$\left\{ \begin{matrix} \text{alg's satisfying} \\ \text{domain hyp.} \end{matrix} \right\} \bigcap \left\{ \begin{matrix} \text{alg's satisfying} \\ \text{target hyp.} \end{matrix} \right\} \bigcap \left\{ \begin{matrix} A : \mathrm{id}_A \text{ satisfies} \\ \text{morphism hyp.} \end{matrix} \right\}$$

are precisely those we wanted to classify on first slide.

$\Rightarrow$ Classification of algebras via $\mathrm{inv}(-)$.
Can deduce classification of algebras via $\mathrm{Ell}(-)$.

Future goal: move more hypotheses (regularity?) to the morphisms, allowing even more general algebras.

a quick application (due to c. schafhauser)

Γ : amenable group; τ : canonical trace on $C_r^*(\Gamma)$

- Higson-Kasparov: Γ satisfies Baum-Connes.
- Lück: range of $K_0(\tau)$ is contained in $\mathbb{Q} \cong K_0(\mathcal{Q})$.
- Tu: $C_r^*(\Gamma)$ satisfies UCT.

Not too hard to produce compatible triple $\operatorname{inv}\left(C_r^*(\Gamma)\right) \to \operatorname{inv}(\mathcal{Q})$ from this data.

Γ : amenable group; $\quad \tau$: canonical trace on $C_r^*(\Gamma)$

- Higson-Kasparov: Γ satisfies Baum-Connes.
- Lück: range of $K_0(\tau)$ is contained in $\mathbb{Q} \cong K_0(\mathcal{Q})$.
- Tu: $C_r^*(\Gamma)$ satisfies UCT.

Not too hard to produce compatible triple $\mathrm{inv}\left(C_r^*(\Gamma)\right) \to \mathrm{inv}(\mathcal{Q})$ from this data.

Punchline:

$\exists$ trace-preserving $C_r^*(\Gamma) \hookrightarrow \mathcal{Q}$ $\Leftrightarrow$ Γ is amenable

A QUICK APPLICATION (DUE TO C. SCHAFHAUSER)

Γ : amenable group; τ : canonical trace on $C_r^*(\Gamma)$

- Higson-Kasparov: Γ satisfies Baum-Connes.
- Lück: range of $K_0(\tau)$ is contained in $\mathbb{Q} \cong K_0(\mathcal{Q})$.
- Tu: $C_r^*(\Gamma)$ satisfies UCT.

Not too hard to produce compatible triple $\operatorname{inv}\left(C_r^*(\Gamma)\right) \to \operatorname{inv}(\mathcal{Q})$ from this data.

Punchline:

$\exists$ trace-preserving $C_r^*(\Gamma) \hookrightarrow \mathcal{Q}$ $\Leftrightarrow$ Γ is amenable $\Leftrightarrow$ $\exists$ trace-preserving $L\Gamma \hookrightarrow \mathcal{R}$

STRATEGY

Define $B_\infty := \ell^\infty B / c_0 B$.

Will prove classification of maps $A \to B_\infty$.

The *trace-kernel ideal* is

$$J_B := \left\{ (x_n) \in B_\infty : \lim_{n\to\infty} \|x_n\|_{2,u} = 0 \right\},$$

The trace-kernel extension

Define $B_\infty := \ell^\infty B / c_0 B$.

Will prove classification of maps $A \to B_\infty$.

The *trace-kernel ideal* is

$$J_B := \left\{ (x_n) \in B_\infty : \lim_{n\to\infty} \|x_n\|_{2,u} = 0 \right\},$$

$$\text{where } \|x\|_{2,u} = \sup_{\tau \in T(B)} \tau(x^*x)^{1/2}.$$

Define $B_\infty := \ell^\infty B/c_0 B$.

Will prove classification of maps $A \to B_\infty$.

The *trace-kernel ideal* is

$$J_B := \left\{(x_n) \in B_\infty : \lim_{n\to\infty} \|x_n\|_{2,u} = 0\right\},$$

$$\text{where } \|x\|_{2,u} = \sup_{\tau\in T(B)} \tau(x^*x)^{1/2}.$$

The *trace-kernel extension* is

$$0 \to J_B \to B_\infty \to B^\infty \to 0.$$

Define $B_\infty := \ell^\infty B / c_0 B$.

Will prove classification of maps $A \to B_\infty$.

The *trace-kernel ideal* is

$$J_B := \left\{ (x_n) \in B_\infty : \lim_{n\to\infty} \|x_n\|_{2,u} = 0 \right\},$$

$$\text{where } \|x\|_{2,u} = \sup_{\tau \in T(B)} \tau(x^* x)^{1/2}.$$

The *trace-kernel extension* is

$$0 \to J_B \to B_\infty \to B^\infty \to 0.$$

Conditions on B $\rightsquigarrow$ J_B nice enough to employ KK machinery

THE TRACE-KERNEL EXTENSION

Define $B_\infty := \ell^\infty B / c_0 B$.

Will prove classification of maps $A \to B_\infty$.

The *trace-kernel ideal* is

$$J_B := \left\{ (x_n) \in B_\infty : \lim_{n\to\infty} \|x_n\|_{2,u} = 0 \right\},$$

$$\text{where } \|x\|_{2,u} = \sup_{\tau \in T(B)} \tau(x^*x)^{1/2}.$$

The *trace-kernel extension* is

$$0 \to J_B \to B_\infty \to B^\infty \to 0.$$

Conditions on B $\rightsquigarrow$ J_B nice enough to employ KK machinery

$\rightsquigarrow$ B^∞ behaves like a finite vNa (sort of)

$$A$$

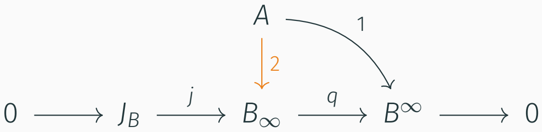
$$0 \longrightarrow J_B \xrightarrow{j} B_\infty \xrightarrow{q} B^\infty \longrightarrow 0$$

APPROXIMATE CLASSIFICATION OF MORPHISMS: MAJOR STEPS

$$\begin{array}{ccccccccc} & & & & A & \overset{1}{\searrow} & & & \\ 0 & \longrightarrow & J_B & \xrightarrow{j} & B_\infty & \xrightarrow{q} & B^\infty & \longrightarrow & 0 \end{array}$$

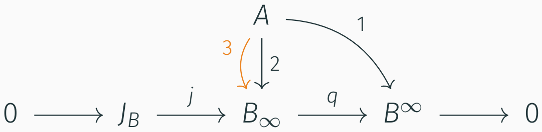
1. Classify morphisms into B^∞
   (von Neumann techniques)

APPROXIMATE CLASSIFICATION OF MORPHISMS: MAJOR STEPS



1. Classify morphisms into B^∞
   (von Neumann techniques)
2. Classify lifts of morphisms to B_∞
   (Ext, *KK* techniques)

APPROXIMATE CLASSIFICATION OF MORPHISMS: MAJOR STEPS



1. Classify morphisms into B^∞
   (von Neumann techniques)
2. Classify lifts of morphisms to B_∞
   (Ext, *KK* techniques)
3. Deal with *KK*-theory, exploiting J_B

Theorem (Castillejos-Evington-Tikuisis-White $+\cdots$)

Let A: exact; B: $\mathcal{Z}$-stable, $T(B) \neq \varnothing$ & compact,
$\mathfrak{t}\colon T(B^\infty) \to T_{\text{amenable}}(A)$ continuous affine.

$$\implies \exists \text{ nuclear } {}^*\text{-hom. } \theta\colon A \to B^\infty$$
$$\text{s.t. } \tau\circ\theta = \mathfrak{t}(\tau) \ \ \forall\, \tau \in T(B^\infty).$$

θ is unique up to unitary equivalence.

Compare with classifying maps from nuclear C^*-alg's into II_1 factors.
In fact: if $T(B) = \{\tau\}$, B^∞ is essentially a vNA ultrapower of $\pi_\tau(B)''$.

Setup: A: sep. exact; B: $\mathcal{Z}$-stable, strict comp. w.r.t. $T(B)$, $T(B) \neq \varnothing$ & compact

Theorem (Existence for lifts)

*$\theta\colon A \to B^\infty$ full nuclear *-hom;*

$\kappa \in KK(A, B_\infty), \quad [q]_{KK}\kappa = [\theta]_{KK}$

$\implies \exists$ *full nuclear lift* $\varphi\colon A \to B_\infty$ *of* θ *s.t.* $[\varphi]_{KK} = \kappa$.

Setup: A: sep. exact; B: $\mathcal{Z}$-stable, strict comp. w.r.t. $T(B)$, $T(B) \neq \varnothing$ & compact

Theorem (Existence for lifts)

*$\theta\colon A \to B^\infty$ full nuclear *-hom;*

$\kappa \in KK(A, B_\infty)$, $\quad [q]_{KK}\kappa = [\theta]_{KK}$

$\implies \exists$ *full nuclear lift* $\varphi\colon A \to B_\infty$ *of* θ *s.t.* $[\varphi]_{KK} = \kappa$.

(Very) roughly:

- θ determines a pullback extension e_θ whose class in $\mathrm{Ext}(A, J_B)$ vanishes.

2. Classifying lifts (a glimpse)

Setup: A: sep. exact; B: $\mathcal{Z}$-stable, strict comp. w.r.t. $T(B)$, $T(B) \neq \varnothing$ & compact

Theorem (Existence for lifts)

*$\theta\colon A \to B^\infty$ full nuclear *-hom;*

$\kappa \in KK(A, B_\infty)$, $\quad [q]_{KK}\kappa = [\theta]_{KK}$

$\quad \implies \exists$ *full nuclear lift* $\varphi\colon A \to B_\infty$ *of* θ

$\qquad$ *s.t.* $[\varphi]_{KK} = \kappa$.

(Very) roughly:

- θ determines a pullback extension e_θ whose class in $\mathrm{Ext}(A, J_B)$ vanishes.
- $[e_\theta] = 0 \implies e_\theta \oplus$(trivial extension) $\approx$ a split extension.

2. CLASSIFYING LIFTS (A GLIMPSE)

Setup: A: sep. exact; B: $\mathcal{Z}$-stable, strict comp. w.r.t. $T(B)$, $T(B) \neq \varnothing$ & compact

Theorem (Existence for lifts)

*$\theta\colon A \to B^\infty$ full nuclear *-hom;*

$\kappa \in KK(A, B_\infty)$, $\quad [q]_{KK}\kappa = [\theta]_{KK}$

$\implies \exists$ *full nuclear lift* $\varphi\colon A \to B_\infty$ *of* θ

s.t. $[\varphi]_{KK} = \kappa$.

(Very) roughly:

- θ determines a pullback extension e_θ whose class in $\mathrm{Ext}(A, J_B)$ vanishes.
- $[e_\theta] = 0 \implies e_\theta \oplus$(trivial extension) $\approx$ a split extension.
- Weyl-von Neumann-Voiculescu type absorption theorems $\implies e_\theta \oplus$(trivial extension) $\approx e_\theta$.

What if we have two (nuclear) lifts φ and ψ of θ?

$$\begin{array}{ccccccccc} & & & & A & & & & \\ & & & & \psi \downarrow\downarrow \varphi & \overset{\theta}{\searrow} & & & \\ 0 & \longrightarrow & J_B & \longrightarrow & B_\infty & \longrightarrow & B^\infty & \longrightarrow & 0 \end{array}$$

Under these hypotheses:

- get class $[\varphi, \psi] \in KK(A, J_B)$
- $[\varphi, \psi]$ determines when φ, ψ are approx. unitarily equiv.

Think of Voiculescu's Theorem:

$$
\begin{array}{ccccccccc}
 & & & & A & \xrightarrow{\quad\theta\quad} & & & \\
 & & & & {\scriptstyle\psi}\downarrow\,\downarrow{\scriptstyle\varphi} & & \searrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{B}(\mathcal{H}) & \longrightarrow & \mathcal{Q}(\mathcal{H}) & \longrightarrow & 0
\end{array}
$$

If φ, ψ are "admissible" (faithful, nondegenerate, and $\varphi(A) \cap \mathcal{K} = \{0\} = \psi(A) \cap \mathcal{K}$), then $\varphi \approx_u \psi$.

Think of Voiculescu's Theorem:

$$\begin{array}{ccccccccc}
 & & & & A & \overset{\theta}{\searrow} & & & \\
 & & & & \psi\downarrow\downarrow\varphi & & & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{B}(\mathcal{H}) & \longrightarrow & \mathcal{Q}(\mathcal{H}) & \longrightarrow & 0
\end{array}$$

If φ, ψ are "admissible" (faithful, nondegenerate, and $\varphi(A) \cap \mathcal{K} = \{0\} = \psi(A) \cap \mathcal{K}$), then $\varphi \approx_u \psi$.

More can be said:

Theorem (Dadarlat-Eilers '01)

Suppose: $\varphi, \psi \colon A \to \mathcal{B}(\mathcal{H})$ *are admissible lifts of* θ.

Then:

$[\varphi, \psi] = 0 \in KK(A, \mathcal{K}) \implies \varphi \approx_u \psi$ *via unitaries in* $\mathcal{K} + \mathbb{C}1_{\mathcal{H}}$.

$$\begin{array}{ccccccccc} & & & & A & & & & \\ & & & & \psi \downarrow\downarrow \varphi & \searrow^{\theta} & & & \\ 0 & \longrightarrow & J_B & \xrightarrow{j} & B_\infty & \xrightarrow{q} & B^\infty & \longrightarrow & 0 \end{array}$$

Theorem (Uniqueness for lifts; CGSTW)

- *A: sep. exact;*
- *B: $\mathcal{Z}$-stable, strict comparison w.r.t. $T(B)$, $T(B) \neq \varnothing$ & cpt;*
- *φ, ψ: unitizably full, nuclear lifts of θ.*

$[\varphi, \psi] = 0 \in KK(A, J_B) \quad \Longrightarrow \quad \varphi \approx_u \psi$ *via unitaries in* $\widetilde{J_B}$.

3. From KK to inv: rotation maps

Consider maps

$$\begin{array}{ccccccccc} & & & & A & & & & \\ & & & & \psi\downarrow\downarrow\varphi & & & & \\ 0 & \longrightarrow & J_B & \xrightarrow{j} & B_\infty & \xrightarrow{q} & B^\infty & \longrightarrow & 0 \end{array}$$

agreeing modulo J_B.

Solutions to lifting problems were encoded in certain KK classes. Want to compute this information in terms inv$(-)$.

For instance, can inv detect when $[\varphi, \psi] \in KK(A, J_B)$ vanishes?

∃ morphism

$$j_*\colon KK(A, J_B) \to \mathrm{Hom}_\Lambda\left(\underline{K}(A), \underline{K}(B_\infty)\right)$$

$$[\varphi, \psi] \mapsto \underline{K}(\varphi) - \underline{K}(\psi)$$

induced by $j\colon J_B \to B_\infty$.

$\exists$ morphism

$$j_*\colon KK(A, J_B) \to \mathrm{Hom}_\Lambda\left(\underline{K}(A), \underline{K}(B_\infty)\right)$$

$$[\varphi, \psi] \mapsto \underline{K}(\varphi) - \underline{K}(\psi)$$

induced by $j\colon J_B \to B_\infty$.

$\underline{K}(\varphi) = \underline{K}(\psi)$ doesn't guarantee $[\varphi, \psi] = 0$.

Subtle obstruction:

Even if $\varphi(u) \sim_h \psi(u)$ is true for $u \in U(A)$, the path ξ connecting them might have nonzero "winding number".

∃ morphism

$$j_*\colon KK(A, J_B) \to \mathrm{Hom}_\Lambda\left(\underline{K}(A), \underline{K}(B_\infty)\right)$$

$$[\varphi, \psi] \mapsto \underline{K}(\varphi) - \underline{K}(\psi)$$

induced by $j\colon J_B \to B_\infty$.

$\underline{K}(\varphi) = \underline{K}(\psi)$ doesn't guarantee $[\varphi, \psi] = 0$.

Subtle obstruction:

Even if $\varphi(u) \sim_h \psi(u)$ is true for $u \in U(A)$, the path ξ connecting them might have nonzero "winding number".

$\rightsquigarrow$ rotation map $R\big([\varphi, \psi]\big)$; (roughly) assigns the function

$$\tau \mapsto \frac{1}{2\pi i}\int_0^1 \tau\Big(\frac{d\xi(t)}{dt}\xi(t)^{-1}\Big)\, dt$$

to $[u] \in K_1(A)$. This is an element of $\mathrm{Aff}\, T(B_\infty)$.

Want to encode this additional obstruction to $[\varphi, \psi] = 0$ using $\overline{K}_1^{\mathrm{alg}}$.

Idea: use Thomsen's map

$$\operatorname{Aff} T(B_\infty) \to \overline{K}_1^{\mathrm{alg}}(B_\infty)$$

which, given $h = h^* \in B_\infty$, maps $\hat{h}$ to $[e^{2\pi i h}]_{\mathrm{alg}}$.

Want to encode this additional obstruction to $[\varphi, \psi] = 0$ using $\overline{K}_1^{\mathrm{alg}}$.

Idea: use Thomsen's map

$$\operatorname{Aff} T(B_\infty) \to \overline{K}_1^{\mathrm{alg}}(B_\infty)$$

which, given $h = h^* \in B_\infty$, maps $\hat{h}$ to $[e^{2\pi i h}]_{\mathrm{alg}}$.

Punchline

Assuming $\underline{K}(\varphi) = \underline{K}(\psi)$,

$$\overline{K}_1^{\mathrm{alg}}(\varphi) - \overline{K}_1^{\mathrm{alg}}(\psi) = 0 \quad \Longrightarrow \quad R\big([\varphi, \psi]\big) = 0 \quad \Longrightarrow \quad [\varphi, \psi] = 0.$$

This let us access the classification theorem for lifts.

SUMMARY

Using inv(—) can provide ∃ and ! for *-hom's

sep., simple, nuclear, UCT	$\longrightarrow$	sep., simple, nuclear, finite, $\mathcal{Z}$-stable

(can do this much more generally!)

Using inv(—) can provide ∃ and ! for *-hom's

sep., simple, nuclear, UCT	$\longrightarrow$	sep., simple, nuclear, finite, $\mathcal{Z}$-stable

(can do this much more generally!)

Can use this to classify algebras
... even in the non-unital setting.

Theorem

Suppose A and B are non-unital, simple, separable, nuclear, $\mathcal{Z}$-stable C^*-algebras satisfying the UCT.

Any isomorphism $\mathrm{Ell}(A) \xrightarrow{\sim} \mathrm{Ell}(B)$ *lifts to an isomorphism* $A \xrightarrow{\sim} B$.

Thank you!